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# Unsteady free convection from a heated sphere at high Grashof number

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## Summary

The free-convective boundary-layer flow over the surface of a sphere whose temperature is suddenly raised to a value greater than its surroundings is considered. Numerical solutions of the boundary-layer equations are presented which give a complete description of the flow and which confirm the appearance of a singularity in the solution at the upper pole after a finite time.

## 1. Introduction

There has been considerable interest shown recently in the free convection boundary-layer flow at the surface of an isolated sphere which is maintained at a temperature different from that of its surroundings. Potter and Riley [1] consider the case of a steady flow and, in particular, they draw attention to the singularity which develops in the solution at the upper pole of the sphere. Brown and Simpson [2] shed further light upon the structure of this singularity and, in addition, address themselves to the unsteady flow which arises when the sphere temperature is suddenly raised above that of its surroundings. For this unsteady problem they argue, on the basis of a local solution in the neighbourhood of the upper pole, that the boundary-layer solution breaks down at a finite time following the initiation of the motion. From a detailed analysis of the complicated multi-layered structure of this breakdown, and a numerical solution of the local governing equations, they are able to estimate the time at which the boundary-layer solution fails. Physically this breakdown corresponds to an eruption of the surface boundary layer to form the free-convective plume above the sphere.

In the present paper we turn attention once more to the unsteady problem and solve completely the unsteady boundary-layer equations, appropriate to the situation which arises when the temperature of the sphere is suddenly raised to a uniform value greater than its surroundings, by numerical methods. It proves convenient to first construct, by standard methods, a series solution which is valid for small time over the surface of the sphere. The solution is then continued to larger times by a finite-difference method proposed by Hall [3]. Except at the upper pole it is possible to monitor the development of the solution from initiation to a steady state. At the upper pole itself the solution develops a singularity at finite time, as in the local solution of Brown and Simpson [2].

In Section 2 we derive the unsteady boundary-layer equations for our high-Grashof number flow and outline the numerical method of solution of these in Section 3. In

Section 4 we present the results of our calculations for a representative case. Principal amongst these is the solution at the upper pole where, within the framework of our global solution, we find precisely the same singular behaviour as in [2]. We conclude that there is now a complete understanding of the nature of the free-convection boundary-layer flow over the surface of a heated sphere.

## 2. Equations of motion

For a Boussinesq fluid, in which variable fluid properties are ignored except in the buoyancy term, the equations which govern a buoyancy-driven, unsteady laminar motion are

$$\nabla \cdot \boldsymbol{v} = 0,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{1}{\rho_\infty} \nabla p + \nu_\infty \nabla^2 \boldsymbol{v} - \frac{\rho - \rho_\infty}{\rho_\infty} \boldsymbol{g}, \tag{2.1}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \kappa_\infty \nabla^2 T.$$

In these equations $p$ denotes pressure, $T$ temperature, $\rho$ density, $\nu$ kinematic viscosity and $\kappa$ thermal diffusivity with a subscript $\infty$ denoting conditions in some ambient or reference state. The vectors $\boldsymbol{v}$, $\boldsymbol{g}$ denote velocity, and acceleration due to gravity respectively. In our problem a sphere, of radius $a$, is at temperature $T_\infty$ in a fluid which is otherwise at rest. At time $t = 0$ the surface temperature of the sphere is raised to, and maintained at, a temperature $T_w > T_\infty$ and a fluid motion ensues. Dimensionless variables are introduced into (2.1) with $a$ as a typical length, $\{ga(T_w - T_\infty)/T_\infty\}^{1/2}$ a typical velocity, $\{aT_\infty/g(T_w - T_\infty)\}^{1/2}$ a typical time and a dimensionless temperature $\theta$ is defined as $\theta = (T - T_\infty)/(T_w - T_\infty)$. In the boundary-layer approximation, which we make below, the pressure is uniform everywhere at leading order and the equation of state reduces to

$$\rho T = \rho_\infty T_\infty. \tag{2.2}$$

We define a Grashof number as $\mathrm{Gr} = ga^3(T_w - T_\infty)/\nu_\infty^2 T_\infty$ and for $\mathrm{Gr} \gg 1$ we introduce a small parameter $\epsilon$ such that $\epsilon^2 = \mathrm{Gr}^{-1}$. As boundary-layer co-ordinates we let $x$ measure angular distance from the lower pole of the sphere and define $y = (r - a)/\epsilon^{1/2} a$ where $r$ is measured radially from the centre of the sphere. If $(U, V)$ are the dimensionless velocity components in the directions of $(x, r)$ increasing we define $u = U$, $v = \epsilon^{1/2} V$ so that finally, in the formal limit $\epsilon \to 0$, we have from (2.1) the following boundary-layer equations to solve

$$\frac{\partial}{\partial x}(u \sin x) + \frac{\partial}{\partial y}(v \sin x) = 0, \tag{2.3a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \theta \sin x, \tag{2.3b}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2}, \tag{2.3c}$$

where $\sigma = \nu_\infty/\kappa_\infty$ is the Prandtl number. The boundary conditions to which (2.3) are subject, for our impulsively heated sphere, are as follows

$$\begin{aligned} &u = \theta = 0, \quad t = 0, \quad y > 0; \\ &u = v = 0, \quad \theta = 1, \quad y = 0, \quad t > 0; \\ &u, \quad \theta \to 0 \text{ as } y \to \infty, \quad t > 0. \end{aligned} \tag{2.4}$$

## 3. Solution procedure

We divide our solution procedure for (2.3), subject to the conditions (2.4), into three parts. At the initial instant a vortex sheet, across which the temperature changes discontinuously, is formed at the surface of the sphere and this is subsequently modified both by diffusion and convection. As a consequence it proves convenient to develop a solution in the form of a series for $t \ll 1$. Also since the lower pole, $x = 0$, is a stagnation point of the flow it is necessary to establish the form of the solution in its neighbourhood. Finally the solution is advanced away from the lines $t = 0$, $x \geqslant 0$ and $x = 0$, $t \geqslant 0$ by adapting a method for the solution of the unsteady boundary-layer equations developed by Hall [3] to the present problem. This method is described in detail in Section 3.3 below but basically our strategy is, for each station $x$, to integrate the equations (2.3) in the direction of $t$ increasing until a steady state is achieved, unless the calculation fails due to the presence of a singularity in the solution at a finite value of $t$. Let us next consider the solution in each of its three parts, as described above, in turn.

### *3.1. The solution for $t \ll 1$*

In common with other problems in which diffusion initially dominates the variable $y/t^{1/2}$ is an appropriate one with which to work. Also for this part of the solution procedure it is convenient to work with a stream function $\psi$. Thus if $\psi$ is defined so that

$$u = \frac{1}{\sin x}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{1}{\sin x}\frac{\partial \psi}{\partial x}, \tag{3.1}$$

then Eqn. (2.3a) is satisfied exactly and we have two equations to solve for $\psi$, $\theta$ derived from (2.3b), (2.3c) using (3.1). The solutions of these equations, for small $t$, are written as

$$\begin{aligned} &\psi = 8t^{3/2}\sin^2 x F_0(\eta) + 128 t^{7/2}\sin^2 x \cos x F_1(\eta) + \cdots, \\ &\theta = \theta_0(\eta) + 32t^2\cos x\,\theta_1(\eta) + \ldots, \qquad \eta = y/2t^{1/2}, \end{aligned} \tag{3.2}$$

where we immediately have $\theta_0 = 1 - \operatorname{erf} \sigma^{1/2}\eta$ and $F_0$, $F_1$, $\theta_1$ satisfy the following ordinary differential equations, with a prime denoting differentiation with respect to $\eta$:

$$\begin{aligned} &F_0''' + 2\eta F_0'' - 4F_0' + \theta_0 = 0, \qquad F_0(0) = F_0'(0) = F_0'(\infty) = 0, \\ &\sigma^{-1}\theta_1'' + 2\eta\theta_1' - 8\theta_1 + \theta_0' F_0 = 0, \qquad \theta_1(0) = \theta_1(\infty) = 0, \\ &F_1''' + 2\eta F_1'' - 12F_1' - F_0'^2 + 2F_0F_0'' + 2\theta_1 = 0, \qquad F_1(0) = F_1'(0) = F_1'(\infty) = 0. \end{aligned} \tag{3.3}$$

The equations (3.3) have been solved using a standard collocation method which employs

Chebychev polynomials. As we shall see below the number of terms we have retained in each of the series (3.2) is sufficient to provide a starting solution for our integration, at each $x$-station, in the direction of $t$ increasing.

### 3.2. *The solution for $x \ll 1$*

Close to $x = 0$ the solution may be developed as a series in powers of $x$, for $t > 0$. The leading term of this series provides a boundary condition which is required, along with that for small $t$, if the solution is to be extended to all points of the region $x, y, t > 0$. For the leading term we write

$$u = xf(y, t), \qquad v = g(y, t), \qquad \theta = h(y, t)$$

so that $f$, $g$ and $h$ satisfy, from Eqns. (2.3),

$$\begin{aligned} &2f + \frac{\partial g}{\partial y} = 0, \\ &\frac{\partial f}{\partial t} + f^2 + g\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} + h, \\ &\frac{\partial h}{\partial t} + g\frac{\partial h}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 h}{\partial y^2}, \end{aligned} \tag{3.4}$$

with boundary conditions at some initial time provided by the solution of Section 3.1 together with $f(0, t) = g(0, t) = 0$, $h(0, t) = 1$, $f(\infty, t) = h(\infty, t) = 0$. The method of solution that we have used to solve Eqns. (3.4) with these boundary conditions is a straightforward adaptation of that described below for the three-dimensional calculation.

### 3.3. *The solution procedure for $x$, $t = O(1)$*

To advance the solution away from $x$, $t = 0$ we have solved the equations (2.3) by a finite-difference method based upon that developed by Hall [3]. A triple-suffix notation is used so that, for example, $u_{l,m,n}$ represents the value of $u$ at the pivotal point $(l, m, n)$ which is the point $(l-1)\delta y$ from the boundary $y = 0$, $(m-1)\delta t$ from the initial instant $t_i$ and $(n-1)\delta x$ from the lower stagnation point $x = 0$, where $\delta x$, $\delta y$, $\delta t$ represent the lengths of the sides of the rectangular mesh we have used. All the derivatives in (2.3) are represented by central differences, which means that both terms of the equation (2.3a) are evaluated at the point $(l + \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2})$ whilst all terms in (2.3b, c) are evaluated at the point $(l, m + \frac{1}{2}, n + \frac{1}{2})$. The nonlinear term $u\partial u/\partial x$ in (2.3b) is treated by an iterative process so that in each cycle of the iterative procedure described below only linear equations are to be solved. Thus the approximation used for the $(j+1)$th cycle is

$$\left(u\frac{\partial u}{\partial x}\right)^{j+1} = u^j\left(\frac{\partial u}{\partial x}\right)^{j+1} + u^{j+1}\left(\frac{\partial u}{\partial x}\right)^j - u^j\left(\frac{\partial u}{\partial x}\right)^j. \tag{3.5}$$

Since Eqns. (2.3) are parabolic in both $x$ and $t$ we may advance the solution in the direction of $x$, $t$ increasing in a variety of ways. In the event we have chosen, at each $x$-station around the sphere, to advance the solution in time $t$, up to some value $t_f$, before proceeding to the next $x$-station. Thus values of $u$, $\theta$ are calculated, at all points across the boundary layer, at the station $(m+1, n+1)$ from the previously calculated values at stations $(m, n)$, $(m+1, n)$ and $(m, n+1)$; the values for $n=1$, that is at $x=0$, are given by obtaining the solution of Eqns. (3.4), and the solution at $m=1$ is given by the small-time solution at $t=t_i$ discussed in Section 3.1. As we shall see, although this procedure yields values of $u$, $\theta$ at the grid points the value of the normal velocity $v$ is given at the centre of the mesh, that is at the station $(m+\frac{1}{2}, n+\frac{1}{2})$.

With the overall procedure as described above we now present the discretised version of the partial differential Eqns. (2.3). As a preliminary we define some subsidiary quantities as follows:

$$u_l^s = \tfrac{1}{4}(u_{l,m+1,n} + u_{l,m,n+1} + u_{l,m,n}),$$

$$u_l^n = u_{l,m,n+1} - u_{l,m+1,n} - u_{l,m,n},$$

$$u_l^m = u_{l,m+1,n} - u_{l,m,n+1} - u_{l,m,n},$$

with quantities $\theta_l^s$, $\theta_l^n$ and $\theta_l^m$ similarly defined. Using central difference formulae, as indicated above, the discretised forms of (2.3a), (2.3c) and (2.3b) may be written, respectively, as

$$\begin{aligned} v_{l+1,m+1/2,n+1/2} = v_{l,m+1/2,n+1/2} - \delta y\Big\{&(u_{l+1,m+1,n+1} + u_{l,m+1,n+1}) \\ &\times\left(\frac{1}{4\delta x} + \tfrac{1}{8}\cot x_{n+1/2}\right) \\ &+ (u_{l+1}^n + u_l^n)/4\delta x + \tfrac{1}{2}\cot x_{n+1/2}(u_{l+1}^s + u_l^s)\Big\}, \end{aligned} \tag{3.6a}$$

$$a_l\theta_{l+1,m+1,n+1} + b_l\theta_{l,m+1,n+1} + c_l\theta_{l-1,m+1,n+1} = d_l, \tag{3.6b}$$

$$\alpha_l u_{l+1,m+1,n+1} + \beta_l u_{l,m+1,n+1} + \gamma_l u_{l-1,m+1,n+1} = \delta_l, \tag{3.6c}$$

where the coefficients $a_l$ etc. in (3.6b, c) are defined as

$$a_l = \frac{v_{l,m+1/2,n+1/2}}{8\delta y} - \frac{1}{4\sigma\delta y^2}, \qquad b_l = \frac{1}{2\delta t} + \frac{1}{2\delta x}\left(\tfrac{1}{4}u_{l,m+1,n+1} + u_l^s\right) + \frac{1}{2\sigma\delta y^2},$$

$$c_l = -\frac{v_{l,m+1/2,n+1/2}}{8\delta y} - \frac{1}{4\sigma\delta y^2},$$

$$\begin{aligned} d_l = &\left(\frac{1}{\sigma\delta y^2} - \frac{v_{l,m+1/2,n+1/2}}{2\delta y}\right)\theta_{l+1}^s - \frac{2}{\sigma\delta y^2}\theta_l^s + \left(\frac{1}{\sigma\delta y^2} + \frac{v_{l,m+1/2,n+1/2}}{2\delta y}\right)\theta_{l-1}^s \\ &- \frac{1}{2\delta t}\theta_l^m - \frac{1}{2\delta x}\left(\tfrac{1}{4}u_{l,m+1,n+1} + u_l^s\right)\theta_l^n, \end{aligned}$$

$$\alpha_l = \frac{v_{l,m+1/2,n+1/2}}{8\delta y} - \frac{1}{4\delta y^2},$$

$$\beta_l = \frac{1}{2\delta t} + \frac{1}{2\delta x}\left(\tfrac{1}{2}\tilde{u}_{l,m+1,n+1} + u_l^s + \tfrac{1}{4}u_l^n\right) + \frac{1}{2\delta y^2},$$

$$\gamma_l = -\frac{v_{l,m+1/2,n+1/2}}{8\delta y} - \frac{1}{4\delta y^2},$$

$$\delta_l = \left(\frac{1}{\delta y^2} - \frac{v_{l,m+1/2,n+1/2}}{2\delta y}\right)u_{l+1}^s - \frac{2}{\delta y^2}u_l^s$$

$$+\left(\frac{v_{l,m+1/2,n+1/2}}{2\delta y} + \frac{1}{\delta y^2}\right)u_{l-1}^s - \frac{u_l^m}{2\delta t} - \frac{u_l^s u_l^n}{2\delta x}$$

$$+\sin x_{n+1/2}\left(\theta_l^s + \tfrac{1}{4}\theta_{l,m+1,n+1}\right) + \frac{\tilde{u}_{l,m+1,n+1}^2}{8\delta x},$$

where we have now used a tilde in $\beta_l$, $\delta_l$ to denote a value of $u_{l,m+1,n+1}$ that has been obtained in an earlier iterative cycle than that under consideration. The iterative procedure, to obtain a converged solution at the station $(m+1, n+1)$, is as follows. First we need an initial estimate of $u_{l,m+1,n+1}$ which we obtain by a simple extrapolation from the three neighbouring points as

$$u_{l,m+1,n+1} = u_{l,m,n+1} + u_{l,m+1,n} - u_{l,m,n}.$$

From this initial estimate we calculate $v_{l,m+1/2,n+1/2}$, at the centre of our computation mesh, for all points across the boundary layer, by letting $l = 1$ to $N-1$ in (3.6a) where $(N-1)\delta y$ is the total thickness assumed for the boundary layer. If in Eqn. (3.6b) we now let $l = 2$ to $N-1$ we have, with the coefficients evaluated from the available values of $v_{l,m+1/2,n+1/2}$, $u_{l,m+1,n+1}$, a set of linear equations which may be solved, using Thomas's algorithm, to give us an estimate of $\theta_{l,m+1,n+1}$. We are then in a position to move onto (3.6c), which may similarly be solved, to give an improved estimate for $u_{l,m+1,n+1}$ which in turn can be used to improve the estimate of $v_{l,m+1/2,n+1/2}$ and so on. This procedure differs from that originally introduced by Hall [3] only with the additional step which involves the calculation of the temperature $\theta$. In practice the method works well and yields a converged solution at the new station quite quickly. As a criterion for convergence we calculate, following each iteration, $\Sigma|u^{j+1} - u^j|$, $\Sigma|v^{j+1} - v^j|$, $\Sigma|\theta^{j+1} - \theta^j|$ and deem the solution to be converged when these three quantities are, simultaneously, less than some prescribed quantity $\delta$. The solution may then proceed to the next station.

The solutions which we have obtained by the methods described above are described in the following section.

## 4. Results and discussion

For all the results which we present in this section we have taken the Prandtl number $\sigma = 0.72$ and for the convergence criterion for our solutions we have set $\delta = 10^{-6}$. In addition we have utilised the small-time solution of Section 3.1 as an initial condition applied at $t_i = 0.25$, even though our two-term expansions are seen to be valid for $t \lesssim 1.0$. The grid sizes that we have used are as follows: $\delta x = \pi/31$ for $x \leqslant 29\pi/31$, $\delta x = \pi/62$ for $29\pi/31 < x \leqslant \pi$, $\delta y = 0.2$ and $\delta t = 0.1$. Earlier results obtained for both steady [1] and unsteady [2] flow suggest that these values are adequate to give results to the accuracy that we quote them here. In order to accommodate the rapidly thickening boundary layer as the upper pole is approached we have applied the far-field conditions at $y = y_\infty = 200$. At each $x$-station we have carried our integration from $t = t_i$ out to $t = t_f = 6$. Up to $x = 3\pi/4$ the solution has settled down to a steady state at this value of $t_f$; however, for larger values of $x$, as the upper pole is approached, the solution does not yield its steady state completely until a greater lapse of time has occurred. We should mention that all our steady-state solutions are in close agreement with the results presented by Potter and Riley [1].

A feature of the steady convective flow [1] is the singular behaviour of the boundary-layer solution as the fluid erupts at the upper pole. Brown and Simpson [2] in their work suggest that this singularity does not appear at infinite time, following the impulsive heating of the sphere, but that the unsteady flow becomes singular at $x = \pi$ at a finite time $t_s$. Their work is based on a local expansion about $x = \pi$, the terminal point of the boundary layer as far as the spatial co-ordinate $x$ is concerned. In spite of the parabolic nature of the governing equations (2.3) one might expect that any singular behaviour has its structure revealed by a local analysis. The objects of the present paper include not only an elucidation of the main flow features as they evolve towards a steady state, but also a verification of the nature of the singularity at $x = \pi$, $t = t_s$ as proposed in [2]. Brown and Simpson [2] show that close to the singular point the flow field may be divided into three regions namely an interior region, within which the flow is effectively inviscid, flanked by regions within which viscous terms cannot be ignored. It is within the inviscid interior region that the flow eruption is seen to take place. In estimating $t_s$ Brown and Simpson choose to examine the singularity in $u/(\pi - x)$ by means of a numerical integration along $x = \pi$. This is not a convenient parameter to work with in our calculations although we can confirm the vanishing of $u$ on $x = \pi$, in which respect the flow differs from the steady case described in [1]. Instead we prefer to work with another representative flow parameter, namely the thermal boundary-layer thickness, $\delta_T$, defined as

$$\delta_T = \int_0^\infty \theta \, \mathrm{d}y. \tag{4.1}$$

Before we consider this singular behaviour further we look at other overall features of the unsteady flow development over the surface of the sphere; in particular we present results for $v_\infty = v(x, y_\infty, t)$, the outflow velocity from the boundary layer, the thermal boundary-layer thickness $\delta_T$ and the local heat transfer $\partial\theta/\partial y|_{y=0}$ at the sphere surface.

Consider first the heat transfer coefficient which is shown in Fig. 1. From the solution outlined in Section 3.1 we have, for $\sigma = 0.72$,

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0} = -0.4787t^{-1/2} - 0.0374 \cos x t^{3/2} + \mathrm{O}(t^{7/2}), \tag{4.2}$$

for $t \ll 1$. We see from Fig. 1 how the heat transfer falls from its initially high values to a clearly defined steady state. However one might note that this progression is not

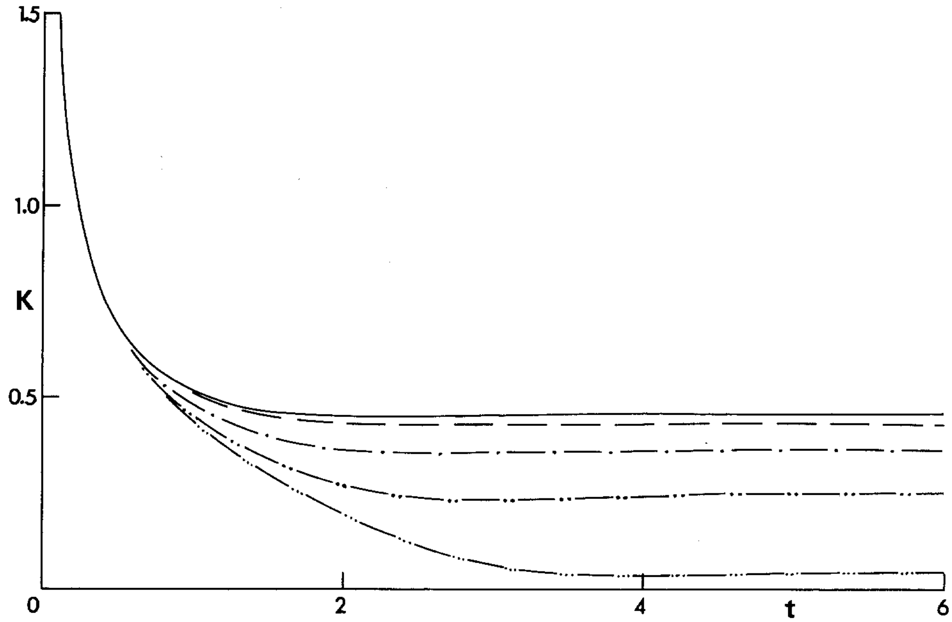


Figure 1. The heat-transfer coefficient $K = -\partial\theta/\partial y|_{y=0}$ at various stations over the surface of the sphere. ——— $x = 0$, — — — — $x = 8\pi/31$, —·—·— $x = 16\pi/31$, —··—··— $x = 23\pi/31$, —···—···— $x = 30\pi/31$.

monotonic; for all values of $x$ the heat transfer exhibits a shallow minimum which appears to be associated with a maximum in the boundary-layer thickness. This "overshoot" in the boundary layer thickness is presumably associated with a delay in the action of convective effects, compared with diffusive effects, at this Prandtl number, at least for $x < \frac{1}{2}\pi$.

In Fig. 2 we show the outflow velocity from the boundary layer, $v_\infty$. From the small-time solution of Section 3.1 we infer, that for $t \ll 1$,

$$v_\infty = -0.9592 \cos x t^{3/2} + 0.0945(2\cos^2 x - \sin^2 x)t^{7/2} + O(t^{11/2}). \tag{4.3}$$

Consider first Fig. 2a in which $v_\infty$ is shown for $x \lesssim 3\pi/4$. For $x \lesssim \frac{1}{2}\pi$ the boundary layer always entrains fluid, and the velocity decreases from zero to its (negative) steady-state value monotonically. On the upper hemisphere, that is for $x > \frac{1}{2}\pi$, a geometrical constraint is removed and as the fluid begins to convect upwards it is seen to have a component of velocity radially outwards. As this transient phenomenon gives way to a steady state there is again entrainment into the boundary layer. We note at this point that for the boundary-layer flow under consideration the steady state is finally characterised by a steady inflow into the boundary layer. Other flow properties may achieve a steady state before the outflow velocity, see for example the heat transfer coefficient in Fig. 1. As we move further around the sphere, see Fig. 2b, so this transient outflow velocity becomes more emphasised until, as $x \to \pi$, there is clear evidence of a singular behaviour appearing in the solution. Since our solution method, described in Section 3, does not yield values of $v$ at the grid points, but at the centre of each mesh, it is not possible for us to comment upon the nature of the singularity from this quantity. The most convenient quantity to examine, in this respect, from our calculations is the thermal boundary-layer thickness $\delta_T$.

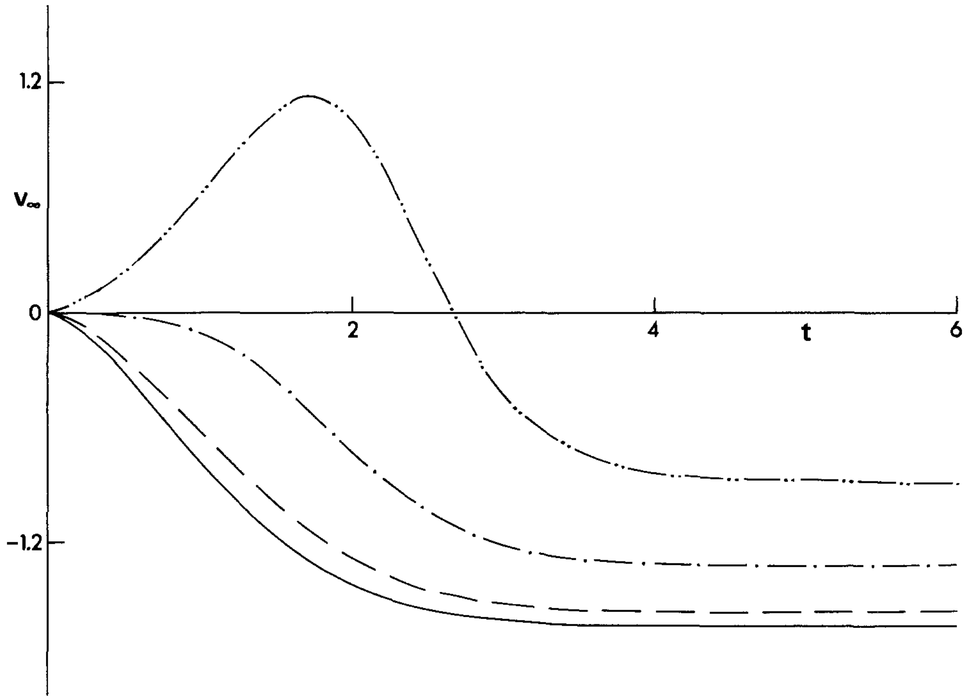


Fig. 2a

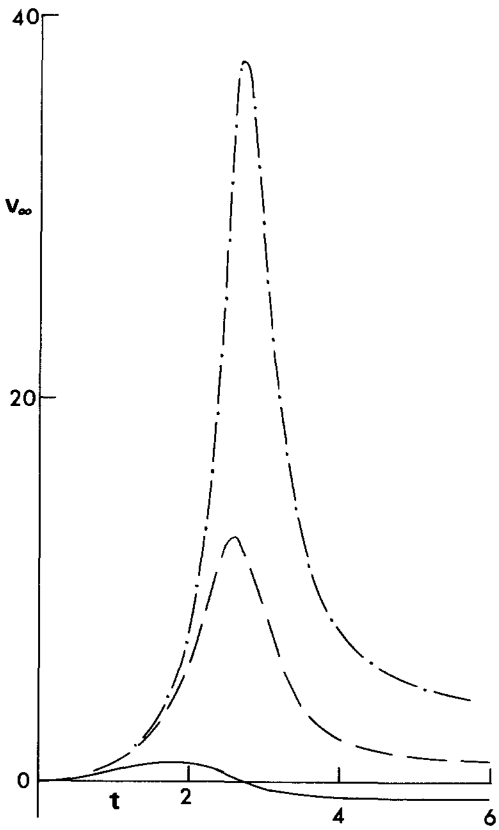


Fig. 2b

Figure 2. The normal velocity, $v_\infty$, at the edge of the boundary layer at various stations $\bar{x} = x + \frac{1}{2}\delta x$: (a) ——— $x = 0$, — — — — $x = 8\pi/31$, –·–·–·– $c = 16\pi/31$, –··–··–··– $x = 23\pi/31$. (b) ——— $x = 23\pi/31$, — — — — $x = 29\pi/31$, –·–·– $x = 30\pi/31$.

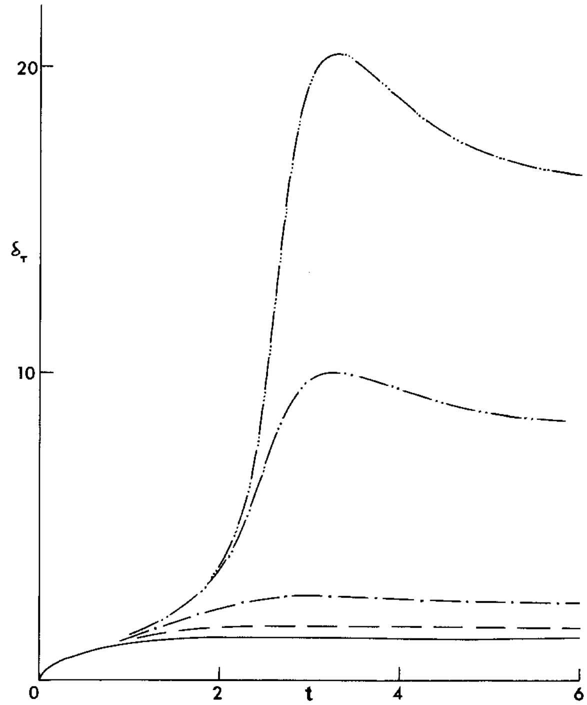


Fig. 3a

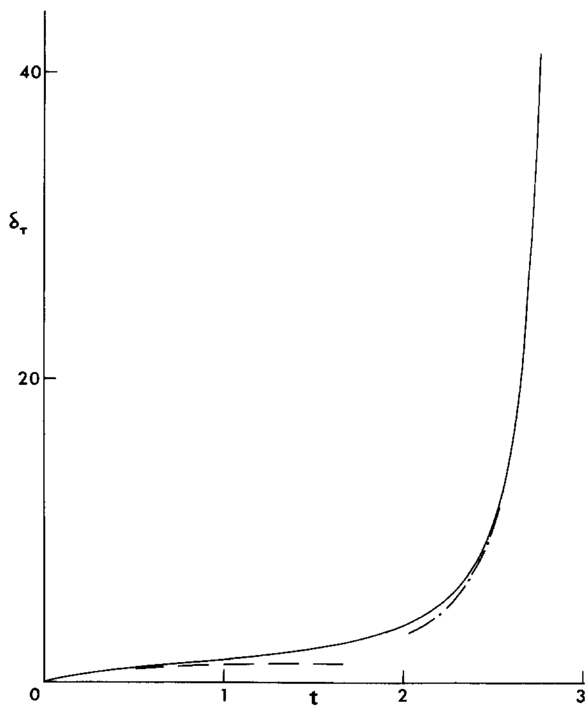


Fig. 3b

Figure 3. The thermal boundary-layer thickness $\delta_T$: (a) ——— $x=0$, — — — — $x=16\pi/31$, –·–·–·– $x=23\pi/31$, –··–··–··– $x=29\pi/31$, –···–···–···– $x=30\pi/31$. (b) ——— $x=\pi$, — — — — the small-time solution (4.4), –·–·–·– the asymptotic solution (4.5).

Figure 3 shows the variation of $\delta_T$, defined in Eqn. (4.1), as a function of $t$. For $t \ll 1$ the solution described in Section 3.1 gives, using (4.1),

$$\delta_T = 1.3298 t^{1/2} - 0.1139 \cos x t^{5/2} + O(t^{9/2}). \tag{4.4}$$

The results obtained in [2] show that as $t \to t_s$ we may expect

$$\delta_T \sim a_0 \tau^{-3/2} + a_1 \tau^{-1}, \tag{4.5}$$

where $\tau = t_s - t$; the constants $t_s$, $a_0$ and $a_1$ are to be determined. In Fig. 3a the variation of $\delta_T$, for values of $x$ which cover almost the whole sphere, is shown and features which have been anticipated earlier are to be seen. Thus, for $x \lesssim 3\pi/4$ the boundary layer remains quite thin and, following a mild overshoot, quickly attains its steady-state value. For $x > 3\pi/4$ a much more rapid thickening of the boundary layer is observed with the overshoot becoming quite pronounced. Indeed for $x$ close to $\pi$ the solution appears to be developing the singular behaviour that was referred to when the outflow velocity $v_\infty$ was under discussion. In Fig. 3b the variation of $\delta_T$ with $t$ along $x = \pi$ is shown, and the singularity which the solution develops is clearly to be observed. Beyond $t = 2.75$ our solutions are no longer accurate, and we have estimated the constants in the asymptotic form (4.5) from the solution up to that point as $a_0 = 3.064$, $a_1 = -0.298$ and $t_s = 2.922$. Both the small-time solution (4.4) and the asymptotic solution (4.5) are also shown in Fig. 3b. The singular time $t_s$ has been estimated by Brown and Simpson [2] following an examination of $u/(\pi - x)$ in their local solution; they estimate $t_s = 2.912$. Considering the diverse methods which have been used to estimate the time at which the solution breaks down at the upper pole the agreement between the values of $t_s$ obtained, differing as they do by $\frac{1}{3}$%, is quite remarkable, and we may conclude that our overall understanding of this unsteady free convection problem is now almost complete; the only gap in our knowledge is along $x = \pi$, $t_s \leqslant t < \infty$ where it is unlikely (see [1]) that the flow is governed by the boundary-layer equations.

## References


[1] J.M. Potter and N. Riley, Free convection from a heated sphere at large Grashof number, *J. Fluid Mech.* 100 (1980) 769–783.

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[3] M.G. Hall, A numerical method for calculating unsteady two-dimensional laminar boundary layers, *Ing. Arch.* 38 (1979) 97–106.